

Elementary functions in Thermodynamic Bethe Ansatz*

J. Suzuki[†]

*Department of Physics, Faculty of Science
Shizuoka University,
Ohya 836, Shizuoka,
Japan*

January 2015

Abstract

Some years ago, Fendley found an explicit solution to the thermodynamic Bethe ansatz (TBA) equation for a $\mathcal{N} = 2$ supersymmetric theory in 2D with a specific F-term. Motivated by this, we seek for explicit solutions for other super-potential cases utilizing the idea from the ODE/IM correspondence. We find that the TBA equations, corresponding to a wider class of super-potentials, admit solutions in terms of elementary functions such as modified Bessel functions and confluent hyper-geometric series.

*Based on talks given at “Infinite Analysis 2014” (Tokyo, Feb. 2014) and at “Integrable lattice models and quantum field theories” (Bad-Honnef, June 2014)

[†]e-mail: sjsuzuk@ipc.shizuoka.ac.jp

1 Introduction

The Thermodynamic Bethe Ansatz (TBA) is one of the most efficient tools in the field of integrable systems [1]. Once input data such as factorized S- matrices [2, 3], special patterns of Bethe ansatz roots (string hypothesis) [4, 5], or the fusion relations [6, 7] are given, it provides finitely or infinitely many coupled integrals equations as output. These equations make the quantitative analyses possible in integrable 1+1D quantum field theories of finite size [2] or 1D quantum systems at finite temperatures [1]. The numerical analysis provides physical quantities such as the specific heat or the magnetic susceptibility for the whole range of temperature [9], or flow of the g function by the change in the system size [8]. On the other hand, some limited information, such as the central charge, is available analytically from the TBA equations. This is due to the fact that the nonlinearity of the TBA equations defies explicit solutions in most of cases.

Some years ago, Fendley [10] obtained a rare example: an explicit solution in the massless limit of an integrable $\mathcal{N} = 2$ supersymmetric theory in 2D. There is a deep structure behind the model which connects the solution to the (massive) TBA equation and the solution to the Painlevé III (PIII) equation. The proof in [10] relies on the heavy machinery on the solution to the PIII equation which has been developed in [11, 12, 13]. Especially it utilizes the Tracy-Widom representation to the PIII solution, valid for a massive theory in general, while the explicit solution in terms of elementary function is possible only in the massless case. Then one may wonder if any simpler derivation is possible for the result in [10], as the massless theory possesses a larger symmetry and thus offers a simpler structure.

In this communication, we will argue that the Ordinary Differential Equation/Integrable Model (ODE/IM) ¹ correspondence provides a much simpler explanation of the solution. This program was actually suggested in [10]. We will make it concrete. The Stokes multipliers τ , associated with a simple ODE with an irregular singularity at infinity, turns out to provide the solution in [10]. This may sound odd as there seems to be no relation between the original problem and the ODE thus there is no reason to consider a specific ODE. There is, however, a relation. We take the super-potential corresponding to Fendley's solution. From the potential, we construct a function, which solves an ODE. At this stage, the Stokes multiplier is a trivial constant. We then “deform” the ODE by a weak gauge field (or small angular momentum). Remarkably, the first nontrivial response of the Stokes multiplier to the gauge field reproduces the solution in [10].

¹For review see [16]

This immediately leads to a generalization. There exists a list of relevant super-potentials [17, 18, 19] for the Landau-Ginzburg description of super-conformal theories. The corresponding TBA equations for perturbed cases are partially derived in [14, 15]. Thus, starting from one of the available potentials, we can construct an ODE and evaluate the first non trivial response to the weak gauge field. The resultant Stokes multipliers are then transformed automatically into Y functions. We will show that these Y functions solve TBA equations in perturbed $\mathcal{N} = 2$ minimal theories with the $SU(2)_k$ and with the $SU(3)_1$ chiral rings, which generalize Fendley's solution for the $SU(2)_1$.

This paper is organized as follows. Section 2 is devoted to a short review on the ODE/IM correspondence. Fendley's solution is re-derived from the Stokes multiplier associated to a special ODE in section 3. In section 4, we apply the working hypothesis obtained in the previous section to the TBA equations for perturbed $\mathcal{N} = 2$ minimal theories in 2D with the $SU(2)_k$ and the $SU(n)_1$ chiral rings. We demonstrate the applications of the exact solutions in section 5. Section 6 is devoted to a summary and future problems.

2 The ODE/IM correspondence

We summarize results from the ODE/IM correspondence which are relevant in the following discussions. For details, see [16].

We consider a simple ODE of n^{th} order in the complex plane $x \in \mathbb{C}$,

$$\left((-1)^{n-1} \frac{d^n}{dx^n} + (x^{n\alpha} - E) \right) \psi(x, E) = 0, \quad (1)$$

where $\alpha \in \mathbb{R}_{\geq -1}$.

Since it has the irregular singularity at ∞ , we conveniently divide the complex plane into sectors. Let \mathcal{S}_j be a sector in the complex plane,

$$\mathcal{S}_j = \left\{ x \mid \left| \arg x - \frac{2j\pi}{n(\alpha+1)} \right| < \frac{\pi}{n(\alpha+1)} \right\}.$$

The sector \mathcal{S}_0 thus includes the positive real axis.

Let $\phi(x, E)$ be a solution to (1) which decays exponentially as x tends to ∞ inside \mathcal{S}_0 ,

$$\frac{d^p \phi(x, E)}{dx^p} \sim (-1)^p \frac{x^{(1-n+2p)\frac{\alpha}{2}}}{\sqrt{n} i^{(n-1)/2}} \exp\left(-\frac{x^{\alpha+1}}{\alpha+1}\right), \quad x \in \mathcal{S}_0 \quad (2)$$

for $p \in \mathbb{Z}_{\geq 0}$.

The crucial observation in [21] is the “discrete rotational symmetry” of (1): the invariance under the simultaneous transformations

$$\begin{aligned} x &\rightarrow q^{-1}x, & E &\rightarrow E\Omega^n, \\ q &= e^{\frac{2\pi i}{n(\alpha+1)}}, & \Omega &= q^{-\alpha}. \end{aligned} \quad (3)$$

We then introduce

$$\phi_j = q^{(n-1)j/2} \phi(q^{-j}x, \Omega^{nj}E). \quad (4)$$

Thanks to the discrete rotational symmetry, any ϕ_j ($j \in \mathbb{Z}$) is a solution to (1). The set $(\phi_j, \dots, \phi_{j+n-1})$ forms the Fundamental System of Solutions (FSS) in \mathcal{S}_j . To see this, we introduce the Wronskian matrix $\Phi_{j_1, \dots, j_m}^{(m)}(x, E)$ and the Wronskian $W_{j_1, \dots, j_m}^{(m)}(x, E) = \det \Phi_{j_1, \dots, j_m}^{(m)}(x, E)$,

$$\Phi_{j_1, \dots, j_m}^{(m)}(x, E) = \begin{pmatrix} \phi_{j_1} & \cdots & \phi_{j_m} \\ \vdots & & \vdots \\ \phi_{j_1}^{(m-1)} & \cdots & \phi_{j_m}^{(m-1)} \end{pmatrix}$$

where $m \leq n$. Especially, when suffixes $\{j\}$ are consecutive integers, e.g., $j_k = j + k - 1$ we write simply $\Phi_j^{(m)}(x, E)$ and their determinants $W_j^{(m)}(x, E)$ (we drop the x dependency when $m = n$). By using the asymptotic form (2), one can check $W_j^{(n)}(E) = 1$, hence the set $(\phi_j, \dots, \phi_{j+n-1})$ is linearly independent.

We are interested in the relation among the FSS in different sectors. Let us start from the relation between \mathcal{S}_0 and \mathcal{S}_1 . Two Wronskian matrices $\Phi_0^{(n)}$ and $\Phi_1^{(n)}$ are simply connected by

$$\Phi_0^{(n)} = \Phi_1^{(n)} \mathcal{M}^{(n)}(E) \quad (5)$$

where

$$\mathcal{M}^{(n)}(E) = \begin{pmatrix} \tau_1^{(1)}(E) & 1 & 0 & \cdots & 0 \\ -\tau_1^{(2)}(E) & 0 & 1 & \cdots & 0 \\ \vdots & & & & 1 \\ (-1)^{n-1} \tau_1^{(n)}(E) & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (6)$$

The entries $\tau_1^{(a)}(E)$ are called the Stokes multipliers. The discrete rotational symmetry then results in

$$\Phi_j^{(n)} = \Phi_{j+1}^{(n)} \mathcal{M}_1^{(n)}(E\Omega^{nj}). \quad (7)$$

The first observation of the ODE/IM correspondence is that this linear relation, evaluated at the origin, can be identified with Baxter's TQ relation [22] for $n = 2$. That is, let

$$\mathbf{T}_1(E) = \tau_1^{(1)}(E\Omega^{-2}), \quad Q^-(E) = \phi(0, E), \quad Q^+(E) = \phi'(0, E). \quad (8)$$

Then one presents $\tau_1 = \phi_0/\phi_1 + \phi_2/\phi_1$ equivalently as

$$\mathbf{T}_1(E) = q^{\mp\frac{1}{2}} \frac{Q^\mp(E\Omega^{-2})}{Q^\mp(E)} + q^{\pm\frac{1}{2}} \frac{Q^\mp(E\Omega^2)}{Q^\mp(E)}. \quad (9)$$

This is known as the Dressed Vacuum Form (DVF) in integrable models.

Suppose that wave functions are given in advance. Then, thanks to the normalization of ϕ , the Stokes multipliers are represented by wave functions, e.g.,

$$\tau_1^{(1)}(E) = W_{0,2,\dots,n}^{(n)}(E). \quad (10)$$

Let us introduce more generally

$$\tau_m^{(a)}(E) = W_{0,\dots,a-1,a+m,\dots,n+m-1}^{(n)}(E). \quad (11)$$

Some of them appear in the connection problem between \mathcal{S}_0 and \mathcal{S}_m [35].

The identity among Wronskians implies

$$\tau_m^{(a)}(E)\tau_m^{(a)}(E\Omega^n) = \tau_m^{(a+1)}(E)\tau_m^{(a-1)}(E\Omega^n) + \tau_{m+1}^{(a)}(E)\tau_{m-1}^{(a)}(E\Omega^n) \quad (12)$$

where $1 \leq a \leq n-1$, $m \in \mathbb{Z}_{\geq 1}$, $\tau_m^{(0)} = \tau_m^{(n)} = 1$ and $\tau_0^{(a)} = 1$.

After a suitable shift of the parameters and a change of variables ($E \rightarrow v, \tau \rightarrow T$), one arrives at the $SU(n)$ T system [29],

$$\mathbf{T}_m^{(a)}(v+i)\mathbf{T}_m^{(a)}(v-i) = \mathbf{T}_m^{(a+1)}(v)\mathbf{T}_m^{(a-1)}(v) + \mathbf{T}_{m+1}^{(a)}(v)\mathbf{T}_{m-1}^{(a)}(v). \quad (13)$$

The conditions $\mathbf{T}_0^{(a)}(v) = \mathbf{T}_m^{(0)}(v) = \mathbf{T}_m^{(n)}(v) = 1$ are again imposed. By employing the further transformation [28, 29],

$$Y_m^{(a)}(v) = \frac{\mathbf{T}_{m-1}^{(a)}(v)\mathbf{T}_{m+1}^{(a)}(v)}{\mathbf{T}_m^{(a+1)}(v)\mathbf{T}_m^{(a-1)}(v)}, \quad (14)$$

one obtains the $SU(n)$ Y system,

$$Y_m^{(a)}(v+i)Y_m^{(a)}(v-i) = \frac{(1+Y_{m-1}^{(a)}(v))(1+Y_{m+1}^{(a)}(v))}{(1+(Y_m^{(a+1)}(v))^{-1})(1+(Y_m^{(a-1)}(v))^{-1})}. \quad (15)$$

The Y system for ADE scattering models was originally introduced in [30].

It is well known under the assumption of the analytic properties on $Y_m^{(a)}$ that the above algebraic equations can be transformed into the TBA equations. This also manifests the ODE/IM correspondence.

We can also formulate the problem on the positive real axis. To simplify notations, let us concentrate on the case $n = 2$ (the radial Schrödinger problem),

$$\left(-\frac{d^2}{dx^2} + (x^{2\alpha} - E) + \frac{\ell(\ell+1)}{x^2}\right)\psi(x, E, \ell) = 0. \quad (16)$$

This is also regarded as the introduction of a gauge field when rewriting it as

$$\left(-\left(\frac{d}{dx} - \frac{\ell}{x}\right)\left(\frac{d}{dx} + \frac{\ell}{x}\right) + (x^{2\alpha} - E)\right)\psi(x, E, \ell) = 0. \quad (17)$$

While, in the absence of the gauge field, the Q function is directly related to the value of the wave function at the origin, as in (8), it is no longer the case with the presence of the gauge field. It is however shown in [23, 21] that the Q function appears naturally if one considers the connection problem of the FSS near the origin and the FSS at large x . Denote two solutions near the origin,

$$\chi^\pm(x, E, \ell) \sim \frac{1}{\sqrt{2\ell+1}} x^{\pm(\ell+\frac{1}{2})+\frac{1}{2}}$$

and set more generally, analogously to (4),

$$\chi_j^\pm(x, E, \ell) = q^{\frac{j}{2}} \chi^\pm(q^{-j}x, \Omega^{2j}E, \ell).$$

The $x \rightarrow 0$ behavior implies

$$\chi_j^\pm(x, E, \ell) = q^{\mp j(\ell+\frac{1}{2})} \chi^\pm(x, E, \ell).$$

The “radial” connection relation is given by ²,

$$\phi(x, E, \ell) = \mathbf{D}^-(E, \ell) \chi^-(x, E, \ell) + \mathbf{D}^+(E, \ell) \chi^+(x, E, \ell), \quad (18)$$

or equivalently,

$$\phi_j(x, E, \ell) = \mathbf{D}^-(E\Omega^{2j}, \ell) \chi_j^-(x, E, \ell) + \mathbf{D}^+(E\Omega^{2j}, \ell) \chi_j^+(x, E, \ell).$$

The connection relations among $\phi_j(x, E, \ell)$ assume the same form, e.g, (5). One then derives the DVF in the radial problem as

$$\mathbf{T}_1(E, \ell) = q^{\mp(\frac{1}{2}+\ell)} \frac{\mathbf{D}^\mp(E\Omega^{-2}, \ell)}{\mathbf{D}^\mp(E, \ell)} + q^{\pm(\frac{1}{2}+\ell)} \frac{\mathbf{D}^\mp(E\Omega^2, \ell)}{\mathbf{D}^\mp(E, \ell)}. \quad (19)$$

²We change the sign of D^+ from [21]

By comparing with (9), one concludes that \mathbf{D}^\pm generalizes Q^\pm for non-zero ℓ case. They recover (9) by putting $\ell = 0$. In terms of \mathbf{D}^\pm , the Wronskian representation of the generalized Stokes multipliers (10) is given by

$$\begin{aligned} \mathbf{T}_j(E, \ell) = & q^{-(j+1)(\ell+\frac{1}{2})} \mathbf{D}^+(E\Omega^{j+1}, \ell) \mathbf{D}^-(E\Omega^{-(j+1)}, \ell) \\ & - q^{(j+1)(\ell+\frac{1}{2})} \mathbf{D}^-(E\Omega^{j+1}, \ell) \mathbf{D}^+(E\Omega^{-(j+1)}, \ell). \end{aligned} \quad (20)$$

This is to be identified with the quantum Wronskian relation [24], except for a difference in normalization as discussed in [21].

3 Revisiting Fendley's solution

In [14, 15], a class of integrable $\mathcal{N}=2$ supersymmetric theories in 2D, described by Landau-Ginzburg actions, has been analyzed. For models with spontaneously broken \mathbf{Z}_n symmetry a set of TBA equations has been proposed. Especially in the latter paper, direct relations of the solution to TBA equations and solutions to PIII or to affine Toda equations are argued.

When the super-potential is given by $W(X) = \frac{X^3}{3} - X$, the explicit TBA equations read,

$$\begin{aligned} A(\theta, \mu) &= 2u(\theta, \mu) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi \cosh(\theta - \theta')} \ln(1 + B(\theta', \mu)^2), \\ B(\theta, \mu) &= - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi \cosh(\theta - \theta')} e^{-A(\theta', \mu)}. \end{aligned} \quad (21)$$

In the above, $u(\theta, \mu) = \mu \cosh \theta$ and μ corresponds to a physical mass. It reduces to $e^\theta/2$ in the massless limit [31].

Fendley found the following explicit solution [10] for the massless case,

$$\begin{aligned} e^{-A(\theta)} &= -2\pi \frac{d}{dz} (\text{Ai}(z))^2, \\ B(\theta) &= 2\pi \frac{d}{dz} \text{Ai}(ze^{i\frac{\pi}{3}}) \text{Ai}(ze^{-i\frac{\pi}{3}}) \end{aligned} \quad (22)$$

where $z = (3e^\theta/4)^{2/3}$.

We will re-derive the solution from the ODE side, starting from,

$$\left(-\frac{d^2}{dx^2} + (x - E) \right) \psi(x, E) = 0 \quad (23)$$

which is the case $(n, \alpha) = (2, \frac{1}{2})$ in (1). It follows from (3) that

$$q = e^{\frac{2\pi}{3}i}, \quad \Omega = e^{-\frac{\pi}{3}i}.$$

It is well known that the Airy function solves this equation. Respecting the leading asymptotic from (2), the desired solution of (23) is given by

$$\phi(x, E) = \sqrt{\frac{2\pi}{i}} \text{Ai}(x - E). \quad (24)$$

This immediately gives Q^{\pm} in (8)

$$\begin{cases} Q^-(E) &= \sqrt{\frac{2\pi}{i}} \text{Ai}(-E), \\ Q^+(E) &= -\sqrt{\frac{2\pi}{i}} \frac{d}{dE} \text{Ai}(-E). \end{cases} \quad (25)$$

We remark that the ODE (23) is not totally independent of the original problem.

Although in the $\mathcal{N} = 2$ symmetric theory, the argument of the superpotential $W(X)$ is a super-field X , we allow for a usual variable in $W(x)$. Then the solution $\phi(x, E)$ has a well known integral representation,

$$\phi(x, E) = \int_{\mathcal{C}} e^{(x-E)\frac{z}{2} W(z(x-E)^{-\frac{1}{2}})} dz. \quad (26)$$

The contour must be chosen so as to reproduce the asymptotic behavior (2) of ϕ . Respecting the “discrete rotational symmetry” and the change of the integration contour, it is easy to show

$$\phi(x, E) + e^{\frac{2\pi}{3}i} \phi(e^{\frac{2\pi}{3}i} x, e^{\frac{2\pi}{3}i} E) + e^{-\frac{2\pi}{3}i} \phi(e^{-\frac{2\pi}{3}i} x, e^{-\frac{2\pi}{3}i} E) = 0. \quad (27)$$

This is equivalent to the three terms relation for the Airy function and it leads to the conclusion $\tau_1 = \mathbf{T}_1 = 1$. We can easily check this by using the DVF(9) and (24). By choosing the upper index in (9) we have

$$\mathbf{T}_1(E) = e^{-\frac{\pi}{3}i} \frac{\text{Ai}(E e^{\frac{2\pi}{3}i})}{\text{Ai}(E)} + e^{\frac{\pi}{3}i} \frac{\text{Ai}(E e^{-\frac{2\pi}{3}i})}{\text{Ai}(E)} = -e^{\frac{2\pi}{3}i} \frac{\text{Ai}(E e^{\frac{2\pi}{3}i})}{\text{Ai}(E)} - e^{-\frac{2\pi}{3}i} \frac{\text{Ai}(E e^{-\frac{2\pi}{3}i})}{\text{Ai}(E)}.$$

Thus $\mathbf{T}_1 = 1$ thanks to (27).

Now T system is trivially represented as

$$\mathbf{T}_1^2 = 1, \quad \mathbf{T}_2 = 0. \quad (28)$$

It simply gives a trivial solution of TBA, $Y_1 = 0$, which is far from Fendley’s solution.

³ Actually, the role played by the Airy function in $\mathcal{N} = 2$ SUSY theory, especially its relation to Q^{\pm} has been firstly noted in [20], independently from [10], exactly in the context of the ODE/IM correspondence.

We then “deform” the ODE by the nonzero angular momentum term as suggested in [10],

$$\left(-\left(\frac{d}{dx} - \frac{\ell}{x}\right)\left(\frac{d}{dx} + \frac{\ell}{x}\right) + x - E\right)\psi(x, E, \ell) = 0. \quad (29)$$

Below we will argue that this replacement leads to the desired T, Y system and to the TBA.

The Stokes multiplier has the form (19),

$$\mathbf{T}_1(E, \ell) = \xi^\mp \frac{\mathbf{D}^\mp(E\Omega^{-2}, \ell)}{\mathbf{D}^\mp(E, \ell)} + \xi^\pm \frac{\mathbf{D}^\mp(E\Omega^2, \ell)}{\mathbf{D}^\mp(E, \ell)}, \quad (30)$$

$$\xi = e^{\frac{\pi-h}{3}i} \quad (31)$$

where $h = -2\ell\pi$. We assume that the following limit exists,

$$\lim_{\ell \rightarrow 0} \frac{1}{\sqrt{2\ell+1}} \mathbf{D}^\pm(E, \ell) = Q^\pm(E). \quad (32)$$

The quantum Wronskian relation is then rewritten with ξ as,

$$\begin{aligned} \mathbf{T}_j(E, \ell) = & \xi^{-(j+1)} \mathbf{D}^+(E\Omega^{j+1}, \ell) \mathbf{D}^-(E\Omega^{-(j+1)}, \ell) \\ & - \xi^{(j+1)} \mathbf{D}^-(E\Omega^{j+1}, \ell) \mathbf{D}^+(E\Omega^{-(j+1)}, \ell). \end{aligned} \quad (33)$$

When q is at a root of unity, the $SU(2)$ T-system (13) closes among finite elements [25]. In the present case, this is due to a simple relation,

$$\mathbf{T}_3(E, \ell) = \xi^3 + \xi^{-3} + \mathbf{T}_1(E, \ell). \quad (34)$$

Then one ends up with

$$\begin{aligned} \mathbf{T}_1(E\Omega, \ell) \mathbf{T}_1(E\Omega^{-1}, \ell) &= 1 + \mathbf{T}_2(E, \ell), \\ \mathbf{T}_2(E\Omega, \ell) \mathbf{T}_2(E\Omega^{-1}, \ell) &= 1 + \mathbf{T}_1(E, \ell) \mathbf{T}_3(E, \ell) \\ &= (\xi^3 + \mathbf{T}_1(E, \ell))(\xi^{-3} + \mathbf{T}_1(E, \ell)). \end{aligned} \quad (35)$$

In the following, we derive (21) from the above truncated T-system as the first nontrivial equation in the expansion of h . Then we will show that a similar expansion of the quantum Wronskian relation (20) yields Fendley’s solution (22).

In this example, the T-system is identified with the Y-system. This is achieved by introducing

$$\mathbf{Y}_t(\theta, \ell) = \mathbf{T}_1(E, \ell), \quad \mathbf{Y}_1(\theta, \ell) = \mathbf{T}_2(E, \ell). \quad (36)$$

Here the parameter θ is related to E by

$$E = E_0 e^{\frac{2}{3}\theta}, \quad (37)$$

and the constant E_0 will be determined later.

The Y-system is represented by new variables as

$$\mathbf{Y}_t(\theta - \frac{\pi}{2}i, \ell) \mathbf{Y}_t(\theta + \frac{\pi}{2}i, \ell) = 1 + \mathbf{Y}_1(\theta, \ell), \quad (38)$$

$$\mathbf{Y}_1(\theta - \frac{\pi}{2}i, \ell) \mathbf{Y}_1(\theta + \frac{\pi}{2}i, \ell) = (\xi^3 + \mathbf{Y}_t(\theta, \ell))(\xi^{-3} + \mathbf{Y}_t(\theta, \ell)). \quad (39)$$

Next we consider the expansion in h . The solution (28), strictly at $h = 0$, suggests the expansions

$$\mathbf{Y}_t(\theta, \ell) = 1 + h y_t(\theta) + O(h^2), \quad \mathbf{Y}_1(\theta, \ell) = h y_1(\theta) + O(h^2). \quad (40)$$

The first nontrivial equation in the expansion of (38) is $O(h)$, while it is $O(h^2)$ for (39),

$$\begin{aligned} y_t(\theta + \frac{\pi}{2}i) + y_t(\theta - \frac{\pi}{2}i) &= y_1(\theta), \\ y_1(\theta + \frac{\pi}{2}i) y_1(\theta - \frac{\pi}{2}i) &= y_t(\theta)^2 + 1. \end{aligned} \quad (41)$$

We set

$$y_t(\theta) = -B(\theta), \quad y_1(\theta) = e^{-A(\theta)} \quad (42)$$

and assume that y_1 and y_t are analytic and nonzero in the strip $\Im m \theta \in [-\pi/2, \pi/2]$. We also assume that the right hand sides of (40) are analytic and nonzero in the narrow strip including the real axis of θ . These assumptions are justified by the solution in (49), a posteriori. One then obtains

$$\begin{aligned} A(\theta) &= m_A e^\theta + C_A - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \ln(1 + B(\theta')^2), \\ B(\theta) &= m_B e^\theta + C_B - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} e^{-A(\theta')} \end{aligned} \quad (43)$$

where “mass terms” are introduced to take account of the zero mode in the Fourier transformation. Without loss of generality we can always choose $m_A = 1$ by tuning the origin of θ (or a redefinition of E_0). It will be later shown that $m_B = 0$. The integration constants C_A, C_B are found to be zero. This can be verified from the asymptotic values $e^{-A(-\infty)} = 2/\sqrt{3}, B(-\infty) = -1/\sqrt{3}$.

We have a remark. The quantum sine Gordon model has $\mathcal{N} = 2$ supersymmetry at a special coupling constant. Fendley et al. [14] utilized this and

started from the TBA for the generic quantum sine Gordon model. Then they took a similar limit in the above and derived (43). Here the initial point is different: we start from the ODE.

The above observation concludes that the expansion of $\mathbf{T}_{1,2}$ in h yields the desired TBA equations (if $m_B = 0$). We then use (33) to obtain explicit solutions, given the data

$$\begin{aligned}\lim_{\ell \rightarrow 0} \frac{1}{\sqrt{2\ell+1}} \mathbf{D}^-(E, \ell) &= Q^-(E) = \sqrt{\frac{2\pi}{i}} \text{Ai}(-E), \\ \lim_{\ell \rightarrow 0} \frac{1}{\sqrt{2\ell+1}} \mathbf{D}^+(E, \ell) &= Q^+(E) = -\sqrt{\frac{2\pi}{i}} \frac{d}{dE} \text{Ai}(-E).\end{aligned}\quad (44)$$

This looks hopeless at first sight, as the expansion of the right hand side of (33) contains derivatives of \mathbf{D}^\pm at $h = 0$ which are unknown to us. By fortunate cancellations of derivative terms, we nevertheless find it possible. First consider the case $j = 0$ in (33) where $\mathbf{T}_0 = 1$. The $O(h^0)$ and the $O(h^1)$ equations read respectively,

$$e^{-\frac{\pi}{3}i} Q^+(E\Omega) Q^-(E\Omega^{-1}) - e^{\frac{\pi}{3}i} Q^-(E\Omega) Q^+(E\Omega^{-1}) = 1, \quad (45)$$

$$\begin{aligned}e^{\frac{\pi}{3}i} \frac{\partial}{\partial h} (D^-(E\Omega, \ell) D^+(E\Omega^{-1}, \ell))|_{h=0} - e^{-\frac{\pi}{3}i} \frac{\partial}{\partial h} (D^+(E\Omega, \ell) D^-(E\Omega^{-1}, \ell))|_{h=0} \\ = \frac{i}{3} \left(e^{\frac{\pi}{3}i} Q^-(E\Omega) Q^+(E\Omega^{-1}) + e^{-\frac{\pi}{3}i} Q^+(E\Omega) Q^-(E\Omega^{-1}) \right).\end{aligned}\quad (46)$$

Next consider $j = 1$ in (33). The $O(h^0)$ term on the right hand side is found to be 1 using (45) (replacing E by $-E$), while the $O(h^1)$ terms contain derivative terms of h . We find that these derivative terms can completely be rewritten in terms of Q^\pm thanks to (46). Altogether, one obtains,

$$\mathbf{T}_1(E) = 1 - i \left(e^{\frac{\pi}{3}i} Q^-(E\Omega) Q^+(E\Omega^{-1}) + e^{-\frac{\pi}{3}i} Q^+(E\Omega) Q^-(E\Omega^{-1}) \right) + O(h^2). \quad (47)$$

Thirdly, take $j = 2$ in (33). It is simplified, as $\Omega^3 = -1$,

$$\begin{aligned}\mathbf{T}_2(E) &= \xi^{-3} D^+(E\Omega^3, \ell) D^-(E\Omega^{-3}, \ell) - \xi^3 D^-(E\Omega^3, \ell) D^+(E\Omega^{-3}, \ell) \\ &= -2ih Q^+(-E) Q^-(-E) + O(h^2).\end{aligned}\quad (48)$$

Then from equations (36), (40), (42), (47) and (48) we conclude

$$\begin{aligned}B(\theta) &= -i \left(e^{\frac{\pi}{3}i} Q^-(Ee^{-\frac{\pi}{3}i}) Q^+(Ee^{\frac{\pi}{3}i}) + e^{-\frac{\pi}{3}i} Q^+(Ee^{-\frac{\pi}{3}i}) Q^-(Ee^{\frac{\pi}{3}i}) \right) \\ &= 2\pi \frac{d}{dE} \text{Ai}(Ee^{-\frac{\pi}{3}i}) \text{Ai}(Ee^{\frac{\pi}{3}i}), \\ e^{-A(\theta)} &= -2i Q^+(-E) Q^-(-E) = -2\pi \frac{d}{dE} \text{Ai}(E)^2\end{aligned}\quad (49)$$

where we use (44).

Finally, let us check $m_B = 0$ in (43) and evaluate E_0 in (37). This is done by evaluating the left hand side of (43) in the limit $\theta \rightarrow \infty$ or equivalently, $E \rightarrow \infty$. The convolution terms do not contribute since the integration kernel becomes exponentially small. One easily evaluates the asymptotic behavior $\Re E \gg 1$ from (49),

$$A(\theta) \sim \frac{4}{3}E^{\frac{3}{2}} \quad B(\theta) \sim -\frac{1}{4}E^{-\frac{3}{2}} + O(E^{-3}).$$

Using them in the second equation of (43), we conclude $m_B = 0$, while the first equation, with the convention $m_A = 1$, leads to

$$\frac{4}{3}E^{\frac{3}{2}} = e^\theta \quad \text{or} \quad E = \left(\frac{3}{4}e^\theta\right)^{\frac{2}{3}}.$$

By identifying z with E in (22), we thus conclude that Fendley's solution is successfully recovered from the ODE.

4 Generalizations

Let us summarize our findings so far. The input is the super-potential $W(x)$. Once this is fixed, we construct a wave function (26) which solves a simple ODE (23). We then “deform” the ODE by the angular momentum term as in (29). This makes the associated Stokes multipliers nontrivial. Then the first non-trivial response of the Stokes multipliers with respect to the small angular momentum yields Fendley's solution.

Since there exists a list of super-potentials for $\mathcal{N} = 2$ supersymmetric theories in 2D [17, 18, 19], one naturally wonders if the above procedure works, starting from other super-potentials. Below we will discuss the super-potential of the type $SU(2)_k$ and $SU(3)_1$ which provide the affirmative evidences to this expectation.

4.1 Exact solution : $SU(2)_k$

In this case, the relevant super-potential takes the form,

$$W_k(x = e^{i\theta} + e^{-i\theta}) = \frac{2}{k+2} \cos(k+2)\theta.$$

More explicitly,

$$\begin{aligned} W_1(x) &= \frac{x^3}{3} - x, & W_2(x) &= \frac{1}{4}(x^4 - 4x^2 + 2), \\ W_3(x) &= \frac{1}{5}(x^5 - 5x^3 + 5x), & W_4(x) &= \frac{1}{6}(x^6 - 6x^4 + 9x^2 - 2) \end{aligned}$$

and so on.

According to the above strategy we first construct a wave function,

$$\phi^{(k)}(x, E) = \int_C e^{(x-E)\frac{k+2}{2}} W_k(z(x-E)^{-1/2}) dz. \quad (50)$$

The contour should meet the requirement that $\phi^{(k)}(x, E)$ is exponentially decreasing on the real axis of x .

Note that (26) is contained as the $k = 1$ case. We immediately see that $\phi^{(k)}(x, E)$ satisfies an ODE,

$$\left(-\frac{d^2}{dx^2} + (x - E)^k\right) \phi^{(k)}(x, E) = 0. \quad (51)$$

As before, we interpret this as the $(n, \alpha) = (2, \frac{1}{2})$ case of a generalized ODE,

$$\left(-\frac{d^2}{dx^2} + (x^{2\alpha} - E)^k\right) \phi^{(k)}(x, E) = 0, \quad (52)$$

with vanishing boundary condition for $\Re x \gg 1$.

This has been proposed to be the ODE for the spin $\frac{k}{2}$ SU(2) case with $q = e^{i\pi/(\alpha k+1)}$ [26, 27].

We have, analogously to (25),

$$Q^-(E) = \sqrt{\frac{2Ei}{(k+2)\pi}} K_{\frac{1}{k+2}}\left(\frac{2}{k+2}(-E)^{\frac{k+2}{2}}\right), \quad (53)$$

$$Q^+(E) = -\frac{d}{dE} Q^-(E) \quad (54)$$

where K_ν stands for the modified Bessel function. We then include the angular momentum term with the effect,

$$Q^-(E) \rightarrow D^-(E, \ell), \quad Q^+(E) \rightarrow D^+(E, \ell).$$

The Stokes multiplier takes the same form as (30), while the parameters take different values,

$$\Omega = e^{-\frac{\pi}{k+2}i}, \quad \xi = e^{\frac{\pi-h}{k+2}i}. \quad (55)$$

The SU(2) T system remains valid, while (34) is replaced by

$$\mathbf{T}_{k+2}(E, \ell) = \xi^{k+2} + \xi^{-(k+2)} + \mathbf{T}_k(E, \ell).$$

The transformation from the T-system to the Y-system is accomplished by [28],

$$\begin{aligned}\mathbf{Y}_j(\theta, \ell) &= \mathbf{T}_{j-1}(E, \ell) \mathbf{T}_{j+1}(E, \ell) \quad (1 \leq j \leq k), \\ \mathbf{Y}_t(\theta, \ell) &= \mathbf{T}_k(E, \ell),\end{aligned}\tag{56}$$

$$E = E_0^{(k)} e^{\frac{2}{k+2}\theta} \tag{57}$$

where \mathbf{T}_0 is set to be 1. The coefficient $E_0^{(k)}$ will be determined later.

We obtain as a result,

$$\begin{aligned}\mathbf{Y}_j(\theta + \frac{\pi}{2}i, \ell) \mathbf{Y}_j(\theta + \frac{\pi}{2}i, \ell) &= (1 + \mathbf{Y}_{j-1}(\theta, \ell))(1 + \mathbf{Y}_{j+1}(\theta, \ell)) \quad (1 \leq j \leq k-1), \\ \mathbf{Y}_k(\theta + \frac{\pi}{2}i, \ell) \mathbf{Y}_k(\theta + \frac{\pi}{2}i, \ell) &= (1 + \mathbf{Y}_{k-1}(\theta, \ell))(1 + \xi^{k+2} \mathbf{Y}_t(\theta, \ell))(1 + \xi^{-(k+2)} \mathbf{Y}_t(\theta, \ell)), \\ \mathbf{Y}_t(\theta + \frac{\pi}{2}i, \ell) \mathbf{Y}_t(\theta + \frac{\pi}{2}i, \ell) &= (1 + \mathbf{Y}_k(\theta, \ell))\end{aligned}\tag{58}$$

where we set $\mathbf{Y}_0 = 0$.

By strictly setting $h = -2\ell\pi = 0$, we obtain constant solutions for $t_{0,j} = \mathbf{T}_j(E, 0)$

$$t_{0,j} = \frac{\sin \frac{(j+1)\pi}{k+2}}{\sin \frac{\pi}{k+2}} \quad 1 \leq j \leq k+1$$

especially $t_{0,k+1} = 0$. Then $\mathbf{Y}_j(\theta, 0)$ is determined by (56). We assume the expansion around $h = 0$,

$$\begin{aligned}\mathbf{T}_j(\theta, \ell) &= t_{0,j} + ht_{1,j}(\theta) + O(h^2) \quad (1 \leq j \leq k+1), \\ \mathbf{Y}_j(\theta, \ell) &= y_{0,j} + hy_{1,j}(\theta) + O(h^2) \quad (1 \leq j \leq k), \\ \mathbf{Y}_t(\theta, \ell) &= 1 + hy_{1,t}(\theta) + O(h^2).\end{aligned}\tag{59}$$

Note that $y_{0,j} \neq 0$ if $j < k$ and $\mathbf{Y}_k(\theta, \ell) = hy_k(\theta)$.

This leads to an important consequence. Although the original Y system (58) consists of $k+1$ equations among $k+1$ Y functions, the first nontrivial relations *close* only among $y_{1,k}(\theta)$ and $y_{1,t}(\theta)$,

$$\begin{aligned}y_{1,k}(\theta + \frac{\pi}{2}i) y_{1,k}(\theta - \frac{\pi}{2}i) &= (t_{0,k-1})^2 (y_{1,t}(\theta)^2 + 1), \\ y_{1,t}(\theta + \frac{\pi}{2}i) + y_{1,t}(\theta - \frac{\pi}{2}i) &= y_{1,k}(\theta).\end{aligned}$$

They are very similar to (41) by identifying $y_{1,k} = y_1$. Consequently, one obtains the analogous TBA

$$\begin{aligned}A(\theta) &= m_A e^\theta - \ln 2 \cos \frac{\pi}{k+2} - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \ln(1 + B(\theta')^2), \\ B(\theta) &= m_B e^\theta - \int \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} e^{-A(\theta')}\end{aligned}\tag{60}$$

where

$$y_{1,t}(\theta) = -B(\theta), \quad y_{1,k}(\theta) = e^{-A(\theta)}. \quad (61)$$

The integration constants are fixed by the asymptotic values,

$$e^{-A(-\infty)} = 2 \cot \frac{\pi}{k+2}, \quad B(-\infty) = -\cot \frac{\pi}{k+2}.$$

As before we choose $E_0^{(k)}$ in (57) such that $m_A = 1$. Below we will argue that $m_B = 0$. Then the resultant TBA agrees with the result in [15] (for $\Theta = \pi/k+2$). We again remark that (60) was derived in [15] by taking a limit from the TBA for the $\mathcal{N} = 0$ sine Gordon model at a special coupling constant which consists of $k+1$ integral equations.

Now we are in position to derive the explicit solutions for $A(\theta)$ and $B(\theta)$ or equivalently, $y_{1,k}(\theta)$ and $y_{1,t}(\theta)$. Thanks to (56) and (59), one immediately finds

$$y_{1,t}(\theta) = t_{1,k}(E), \quad y_{1,k}(\theta) = t_{0,k-1}t_{1,k+1}(E). \quad (62)$$

The right hand side of above equations can be evaluated through the quantum Wronskian relations (33) with $j = 0, k$ and $k+1$. Note that Ω and ξ are given in (55). After simple manipulations, we obtain,

$$\begin{aligned} y_{1,k}(\theta) &= -2it_{0,k-1}Q^-(-E)Q^+(-E), \\ y_{1,t}(\theta) &= -i\left(e^{\frac{\pi}{k+2}i}Q^-(-E\Omega)Q^+(-E\Omega^{-1}) + e^{-\frac{\pi}{k+2}i}Q^-(-E\Omega^{-1})Q^+(-E\Omega)\right). \end{aligned}$$

By the use of (53), solutions are represented explicitly. Let

$$\text{Ai}^{(k)}(E) = \frac{1}{\pi} \sqrt{\frac{E}{k+2}} K_{\frac{1}{k+2}}\left(\frac{2}{k+2}E^{\frac{k+2}{2}}\right) \quad (63)$$

which reduces to the Airy function Ai if $k = 1$. Then one finds solutions which generalize (49) naturally,

$$\begin{aligned} B(\theta) &= 2\pi \frac{d}{dE} \text{Ai}^{(k)}(E\Omega) \text{Ai}^{(k)}(E\Omega^{-1}), \\ e^{-A(\theta)} &= -4\pi \cos \frac{\pi}{k+2} \frac{d}{dE} \text{Ai}^{(k)}(E)^2. \end{aligned} \quad (64)$$

Then we substitute the above explicit solution into (60) and take the limit $\theta \rightarrow \infty$. From the known asymptotic behavior

$$\text{Ai}^{(k)}(E) \sim \frac{E^{-\frac{k}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{k+2}E^{\frac{k+2}{2}}},$$

one easily checks that $m_B = 0, m_A = 1$ and

$$E = E_0^{(k)} e^{\frac{2}{k+2}\theta}, \quad E_0^{(k)} = \left(\frac{k+2}{4}\right)^{\frac{2}{k+2}}.$$

We assume $k \in \mathbb{N}$ in deriving TBA (60). Once it is obtained, however, k enters as a mere parameter. One can take, for example, $k \in \mathbb{R}_{\geq 0}$. We checked numerically that (64) still satisfies (60) in this case.

4.2 Exact solution : $SU(3)_1$

Next consider the perturbation of the $SU(3)_1$ type. The super-potential reads,

$$W(z, x) = \frac{z^4}{4} - xz.$$

According to our working hypothesis, we introduce

$$\phi(x, E) = \int_{\mathcal{C}} e^{W(z, x-E)} dz$$

which satisfies the 3rd order ODE,

$$\left(\frac{d^3}{dx^3} + x - E\right)\phi(x, E) = 0. \quad (65)$$

As always, we choose \mathcal{C} such that the asymptotic behavior of ϕ agrees with (2).

We regards this as a special case of (1), with $n = 3$ and $\alpha = 1/3$. The analysis in [32, 33] shows that the ODE is related to CFT with $A_2^{(2)}$ symmetry.

The explicit solution to (65) with the desired property (2) is given by Meijer's G function or a linear combination of confluent hypergeometric series as discussed in Appendix B. Here we do not specify its explicit form but use a symbol φ :

$$\varphi(x - E) = \phi(x, E). \quad (66)$$

We define, for later use,

$$\begin{aligned} Q^{[0]}(E) &= \varphi(-E), & Q^{[1]}(E) &= -\frac{d}{dE}\varphi(-E), \\ Q^{[2]}(E) &= \frac{1}{2}\frac{d^2}{dE^2}\varphi(-E) \end{aligned} \quad (67)$$

As before, we consider a “radial” ODE [34]

$$\left(\mathcal{D}(g_2 - 2)\mathcal{D}(g_1 - 1)\mathcal{D}(g_0) + x - E\right)\psi(x, E, \mathbf{g}) = 0. \quad (68)$$

The operator $\mathcal{D}(g)$ is defined by

$$\mathcal{D}(g) := \frac{d}{dx} - \frac{g}{x}.$$

The parameters g_i are constrained by

$$g_0 + g_1 + g_2 = 3.$$

We denote by $\phi(x, E, \mathbf{g})$, the solution which behaves as (2) in \mathcal{S}_0 where \mathbf{g} stands for $\{g_0, g_1, g_2\}$. We further introduce

$$\phi_j(x, E, \mathbf{g}) = q^j \phi(q^{-j}x, \Omega^{3j}E, \mathbf{g}) \quad (69)$$

where $q = e^{2\pi i/(3\alpha+3)}$ and $\Omega = q^{-\alpha}$ ⁴. Then any $\phi_j(x, E, \mathbf{g})$, $j \in \mathbb{Z}$ is also a solution to the ODE.

The connection relations among $\{\phi_j\}$ remain formally the same as (5), although the components in (6) now possess a dependency on \mathbf{g} .

We denote by $\{\chi^{[i]}\}$, another FSS of (68) near the origin characterized by the behavior for $x \rightarrow 0$,

$$\chi^{[i]}(x, E, \mathbf{g}) \sim \mathcal{N}_g x^{g_i} \quad (i = 0, 1, 2), \quad \mathcal{N}_g = \left(\prod_{0 \leq j < i \leq 2} (g_i - g_j) \right)^{-\frac{1}{3}}.$$

The ordering, $\Re g_0 < \Re g_1 < \Re g_2$ is assumed from now on. The normalization factor \mathcal{N} is chosen so that the Wronskian determinant of $\{\chi^{[0]}, \chi^{[1]}, \chi^{[2]}\}$ is unity. Following (69), we set

$$\chi_j^{[i]}(x, E, \mathbf{g}) = q^j \chi^{[i]}(q^{-j}x, \Omega^{3j}E, \mathbf{g}) = q^j \chi^{[i]}(q^{-j}x, q^{-j}E, \mathbf{g}).$$

It can be shown as in the case of $\text{SU}(2)$ [21],

$$\chi_j^{[i]}(x, E, \mathbf{g}) = q^{j(1-g_i)} \chi^{[i]}(x, E, \mathbf{g}).$$

We set analogously to (18),

$$\phi(x, E, \mathbf{g}) = \sum_{i=0}^2 D^{[i]}(E, \mathbf{g}) \chi^{[i]}(x, E, \mathbf{g})$$

or slightly more generally,

$$\phi_j(x, E, \mathbf{g}) = \sum_{i=0}^2 D^{[i]}(Eq^{-j}, \mathbf{g}) \chi_j^{[i]}(x, E, \mathbf{g}). \quad (70)$$

⁴We are now considering $\alpha = \frac{1}{3}$ thus explicitly $q = \Omega^{-3} = e^{i\pi/2}$

When $\{g_0, g_1, g_2\} = \{0, 1, 2\}$ the original ODE (65) is recovered. This implies the limit

$$\lim_{\{g_0, g_1, g_2\} \rightarrow \{0, 1, 2\}} \mathcal{N}_g D^{[i]}(E, \mathbf{g}) = Q^{[i]}(E). \quad (71)$$

The generalized Stokes multipliers

$$\tau_m^{(1)}(E, \mathbf{g}) = W_{0, m+1, m+2}^{(3)}(E, \mathbf{g}), \quad \tau_m^{(2)}(E, \mathbf{g}) = W_{0, 1, m+2}^{(3)}(E, \mathbf{g}) \quad (72)$$

are expressible in terms of $D^{[i]}(E, \mathbf{g})$, using (70) in the above,

$$\begin{aligned} \tau_m^{(1)}(E, \mathbf{g}) &= \sum_{\sigma} \text{sgn} \sigma q^{(m+1)(1-g_{\sigma_2})} q^{(m+2)(1-g_{\sigma_3})} \\ &\quad \times D^{[\sigma_1]}(E, \mathbf{g}) D^{[\sigma_2]}(Eq^{-(m+1)}, \mathbf{g}) D^{[\sigma_2]}(Eq^{-(m+2)}, \mathbf{g}), \\ \tau_m^{(2)}(E, \mathbf{g}) &= \sum_{\sigma} \text{sgn} \sigma q^{(1-g_{\sigma_2})} q^{(m+2)(1-g_{\sigma_3})} \\ &\quad \times D^{[\sigma_1]}(E, \mathbf{g}) D^{[\sigma_2]}(Eq^{-1}, \mathbf{g}) D^{[\sigma_2]}(Eq^{-(m+2)}, \mathbf{g}) \end{aligned} \quad (73)$$

where σ signifies the permutation of $\{0, 1, 2\}$.

They also have the DVF representations [34]. The explicit forms are given in Appendix A for $\tau_1^{(1)}$ and $\tau_1^{(2)}$. The results there suggest that it is convenient to set,

$$\mathbf{T}_m^{(a)}(E) = \tau_m^{(a)}(Eq^{\frac{a+m}{2} + \frac{1}{4}}, \mathbf{g}) \quad (74)$$

and

$$E = E_0 e^{\frac{3}{4}\theta}. \quad (75)$$

The \mathbf{g} dependency is dropped in \mathbf{T} . The SU(3) T-system is then recovered,

$$\mathbf{T}_m^{(a)}(\theta + \frac{\pi}{3}i) \mathbf{T}_m^{(a)}(\theta - \frac{\pi}{3}i) = \mathbf{T}_m^{(a+1)}(\theta) \mathbf{T}_m^{(a-1)}(\theta) + \mathbf{T}_{m+1}^{(a)}(\theta) \mathbf{T}_{m-1}^{(a)}(\theta) \quad (76)$$

where $a = 1, 2$ and $m \geq 1$. We set $\mathbf{T}_m^{(0)} = \mathbf{T}_m^{(3)} = 1$ and $\mathbf{T}_0^{(a)} = 1$ and used $q = e^{\frac{\pi}{2}i}$. We perform a transformation similar to (14) [29],

$$\mathbf{Y}_m^{(a)}(\theta) = \frac{\mathbf{T}_{m-1}^{(a)}(\theta) \mathbf{T}_{m+1}^{(a)}(\theta)}{\mathbf{T}_m^{(a-1)}(\theta) \mathbf{T}_m^{(a+1)}(\theta)}. \quad (77)$$

This yields the SU(3) Y system

$$\mathbf{Y}_m^{(a)}(\theta + \frac{\pi}{3}i) \mathbf{Y}_m^{(a)}(\theta - \frac{\pi}{3}i) = \frac{(1 + \mathbf{Y}_{m-1}^{(a)}(\theta))(1 + \mathbf{Y}_{m+1}^{(a)}(\theta))}{(1 + (\mathbf{Y}_m^{(a+1)}(\theta))^{-1})(1 + (\mathbf{Y}_m^{(a-1)}(\theta))^{-1})} \quad (78)$$

where $a = 1, 2$, $m \geq 1$ and set $(\mathbf{Y}_m^{(0)})^{-1} = (\mathbf{Y}_m^{(3)})^{-1} = 0$, $\mathbf{Y}_0^{(a)} = 0$. Note that, contrary to the $\text{SU}(2)$ case, the truncation of the \mathbf{Y} system (78) to a finite set does not occur⁵. From now on, we consider the special choice on g_i ,

$$g_0 = \frac{h}{2\pi}, \quad g_1 = 1 + \frac{h}{2\pi}, \quad g_2 = 2 - \frac{h}{\pi} \quad (79)$$

and take the limit $h \rightarrow 0$.

We can easily check (for $a = 1, 2$, $k \in \mathbb{Z}_{\geq 0}$)

$$\lim_{h \rightarrow 0} \mathbf{T}_{4k}^{(a)} = \lim_{h \rightarrow 0} \mathbf{T}_{4k+1}^{(a)} = 1, \quad \lim_{h \rightarrow 0} \mathbf{T}_{4k+2}^{(a)} = \lim_{h \rightarrow 0} \mathbf{T}_{4k+3}^{(a)} = 0. \quad (80)$$

This motivates us to assume the expansions,

$$\begin{aligned} \mathbf{T}_m^{(a)}(\theta) &= 1 + ht_{1,m}^{(a)}(\theta) + h^2 t_{2,m}^{(a)}(\theta) + O(h^3) & m = 4k, 4k+1, \\ \mathbf{T}_m^{(a)}(\theta) &= ht_{1,m}^{(a)}(\theta) + h^2 t_{2,m}^{(a)}(\theta) + O(h^3) & m = 4k+2, 4k+3. \end{aligned} \quad (81)$$

Similarly consider the expansions of the \mathbf{Y} function,

$$\mathbf{Y}_m^{(a)}(\theta) = y_{0,m}^{(a)} + hy_{1,m}^{(a)}(\theta) + h^2 y_{2,m}^{(a)}(\theta) + O(h^3). \quad (82)$$

By using the $k = 1$ case of (81) in (77) we obtain

$$\mathbf{Y}_1^{(a)}(\theta) = ht_{1,2}^{(a)}(\theta) + h^2(-t_{1,2}^{(a)}(\theta)t_{1,1}^{(\bar{a})}(\theta) + t_{2,2}^{(a)}(\theta)) + O(h^3) \quad (83)$$

for $(a, \bar{a}) = (1, 2)$ or $(2, 1)$. Substituting these into (78), one deduces expansions for other \mathbf{Y} functions,

$$\begin{aligned} \mathbf{Y}_2^{(a)}(\theta) &= -1 + hy_{1,2}^{(a)}(\theta) + h^2 y_{2,2}^{(a)}(\theta) + O(h^3), \\ \mathbf{Y}_3^{(a)}(\theta) &= -1 + hy_{1,3}^{(a)}(\theta) + h^2 y_{2,3}^{(a)}(\theta) + O(h^3) \quad (a = 1, 2) \end{aligned} \quad (84)$$

and so on. There are complex expressions of $y_{j,m}^{(a)}$ in terms of $t_{j',m'}^{(a')}$, which we shall omit.

The first nontrivial relations of the case $m = 1$ in (78) exist at $O(h^2)$,

$$y_{1,1}^{(a)}(\theta + \frac{\pi}{3}i)y_{1,1}^{(a)}(\theta - \frac{\pi}{3}i) = y_{1,1}^{(\bar{a})}(\theta)y_{1,2}^{(a)}(\theta) \quad (85)$$

where $(a, \bar{a}) = (1, 2)$ or $(2, 1)$.

Next consider the $m = 2$ case. The $O(h^0)$ equations require

$$y_{1,3}^{(a)}(\theta) = -y_{1,2}^{(\bar{a})}(\theta) \quad (86)$$

⁵ There is, however, an elaborate way to introduce a set of nonlinear equations which truncate among finite elements [36]. We however do not adopt that approach here.

while the $O(h^1)$ equations yield

$$y_{1,2}^{(a)}(\theta + \frac{\pi}{3}i) + y_{1,2}^{(a)}(\theta - \frac{\pi}{3}i) = -y_{1,1}^{(a)}(\theta) + y_{1,2}^{(\bar{a})}(\theta) + \frac{y_{2,3}^{(a)}(\theta) + y_{2,2}^{(\bar{a})}(\theta)}{y_{1,2}^{(\bar{a})}(\theta)}. \quad (87)$$

This again significantly differs from the $SU(2)$ case. The equations do not close among $y_{1,m}^{(a)}$, i.e., the first order coefficients in h . To determine the last terms in (87), one must consider equations containing $y_{3,m}^{(a)}$ and so on. This leads to an infinite hierarchy of equations.

There are, however, miraculous cancellations. In Appendix C, it will be shown that as a direct consequence of (73), the last terms in (87) are simplified drastically,

$$\frac{y_{2,3}^{(a)}(\theta) + y_{2,2}^{(\bar{a})}(\theta)}{y_{1,2}^{(\bar{a})}(\theta)} = \begin{cases} 3i & a = 1, \\ -3i & a = 2. \end{cases} \quad (88)$$

Thus equations (85) and (87) provide the closed relations among $y_{1,m}^{(a)}$ ($a, m = 1, 2$).

The result can be neatly written down by introducing,

$$y_{1,1}^{(a)}(\theta) = e^{-A_a(\theta)}, \quad y_{1,2}^{(1)}(\theta) = B_0(\theta) + i, \quad y_{1,2}^{(2)}(\theta) = B_{\bar{0}}(\theta) - i. \quad (89)$$

Then we have

$$\begin{aligned} e^{-A_1(\theta + \frac{\pi}{3}i) - A_1(\theta - \frac{\pi}{3}i) + A_2(\theta)} &= B_0(\theta) + i, \\ e^{-A_2(\theta + \frac{\pi}{3}i) - A_2(\theta - \frac{\pi}{3}i) + A_1(\theta)} &= B_{\bar{0}}(\theta) - i, \\ B_0(\theta + \frac{\pi}{3}i) + B_0(\theta - \frac{\pi}{3}i) - B_{\bar{0}}(\theta) &= -e^{-A_1(\theta)}, \\ B_{\bar{0}}(\theta + \frac{\pi}{3}i) + B_{\bar{0}}(\theta - \frac{\pi}{3}i) - B_0(\theta) &= -e^{-A_2(\theta)}. \end{aligned}$$

Under suitable assumptions on analyticity, we obtain the following TBA

$$\begin{aligned} A_r(\theta) &= m_r e^\theta - \sum_{\ell=0, \bar{0}} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi_{r,\ell}(\theta - \theta') \ln(ia_\ell + B_\ell(\theta')) \quad (r = 1, 2), \quad (90) \\ B_\ell(\theta) &= - \sum_{r=1,2} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi_{r,\ell}(\theta - \theta') e^{-A_r(\theta')} \quad (\ell = 0, \bar{0}) \end{aligned}$$

where $a_0 = -a_{\bar{0}} = 1$ and

$$\begin{aligned}\Phi_{1,0}(\theta) &= \Phi_{2,\bar{0}}(\theta) = \frac{\sin \frac{\pi}{3}}{\cosh \theta - \cos \frac{\pi}{3}}, \\ \Phi_{1,\bar{0}}(\theta) &= \Phi_{2,0}(\theta) = \frac{\sin \frac{\pi}{3}}{\cosh \theta + \cos \frac{\pi}{3}}.\end{aligned}$$

We have used the limiting values

$$\begin{aligned}e^{-A_1(-\infty)} &= \frac{3}{2\sqrt{2}}e^{\frac{\pi}{4}i}, & e^{-A_2(-\infty)} &= \frac{3}{2\sqrt{2}}e^{-\frac{\pi}{4}i}, \\ B_0(-\infty) &= -\frac{3+i}{4}, & B_{\bar{0}}(-\infty) &= -\frac{3-i}{4}\end{aligned}$$

to fix the integration constants. The mass coefficients m_r can be set to unity with proper choice of E_0 in (75).

Next let us discuss the solutions in terms of φ in (66). Our strategy is similar to the $SU(2)$ case. Expand the Wronskian relations (72) in powers of h . Use the fact $\tau_0^{(1)} = 1$ to replace the derivatives of $D^{[i]}$ by $Q^{[i]}$ taking (71) into account. Then use (67) and represent the result by φ . There is, of course, no guarantee that all derivatives can be rewritten by this trick. We however found that, parallel to the $SU(2)$ case, this replacement can be done successfully.

As the analogous argument is presented for (88) in Appendix C, we shall omit details and write down the final results,

$$\begin{aligned}e^{-A_1(\theta)} &= t_{1,2}^{(1)}(\theta) = 3\omega^3 w_E[\varphi(E\omega^{-1}), \varphi(E\omega^3)] \frac{d^2}{dE^2} \varphi(E\omega^{-1}), \\ e^{-A_2(\theta)} &= t_{1,2}^{(2)}(\theta) = 3\omega^{-3} w_E[\varphi(E\omega^{-3}), \varphi(E\omega)] \frac{d^2}{dE^2} \varphi(E\omega)\end{aligned}\quad (91)$$

where $\omega = e^{\frac{\pi}{8}i}$ and $w_E[f, g] = f \frac{d}{dE} g - g \frac{d}{dE} f$.

We use (C.97) in Appendix C to write down the solutions for B_0 and $B_{\bar{0}}$,

$$\begin{aligned}B_{\bar{0}}(\theta) &= y_{1,2}^{(2)}(\theta) + i = 3\omega^{-1} w_E[\varphi(E\omega^{-1}), \varphi(E\omega^{-5})] \frac{d^2}{dE^2} \varphi(E\omega^3) + i, \\ B_0(\theta) &= y_{1,2}^{(1)}(\theta) - i = -3\omega w_E[\varphi(E\omega), \varphi(E\omega^5)] \frac{d^2}{dE^2} \varphi(E\omega^{-3}) - i.\end{aligned}\quad (92)$$

By using the $n = 3$ case of (2) in (91) and (92), the $\theta \rightarrow \infty$ asymptotic forms of A_r , B_0 and $B_{\bar{0}}$ are easily derived. Substituting these into (90), we can fix E_0 in (75) (with $m_r = 1$)

$$E_0 = \left(\frac{4}{3\sqrt{3}} \right)^{\frac{3}{4}}.$$

5 Applications

5.1 Analytic evaluation of the Cecotti-Fendley-Intriligator-Vafa Index

We have been able to derive some explicit solutions for $\mathcal{N}=2$ TBA. This may open up the possibility to investigate these systems in a quantitative manner. As an application of the above results, we discuss the analytic evaluation of the Cecotti-Fendley-Intriligator-Vafa (CFIV) index \mathbf{Q}_{CFIV} [15] for the $\text{SU}(2)_k$ case in the massless limit. The CFIV index is defined by

$$\text{Tr}(-1)^F F e^{-\beta H}$$

where F denotes the Fermion number. This is reformulated in the TBA framework and it is expressed as

$$\mathbf{Q}_{\text{CFIV}}(\mu) = \mu \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta e^{-A(\theta, \mu)}$$

where $\mu = m\beta$ and m is the mass. In the view point of tt^* geometry, μ corresponds to the radial coordinate and $\mathbf{Q}_{\text{CFIV}}(\mu)$ is expressible in terms of the solution to the PIII equation. We are interested in the massless limit. In this case \mathbf{Q}_{CFIV} is a constant and given by

$$\mathbf{Q}_{\text{CFIV}} = 2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{\theta} e^{-A(\theta)}$$

where the factor 2 takes account of the contributions from the left and right edges. By change of integration variables from θ to E , we have

$$\begin{aligned} \mathbf{Q}_{\text{CFIV}} &= 2 \int_0^{\infty} dE E^{\frac{k}{2}} e^{-A(\theta)} \\ &= 16\pi \cos \frac{\pi}{k+2} \int_0^{\infty} dE E^{\frac{k}{2}} \text{Ai}^{(k)}(E) \frac{d}{dE} \text{Ai}^{(k)}(E). \end{aligned}$$

Using the recursion relation for the modified Bessel function, $\frac{d}{dE} \text{Ai}^{(k)}(E)$ is given by a sum of three terms. We introduce the notation,

$$\mathcal{G}_n(p, q, \ell) = \int_0^{\infty} dE E^{2n+\ell+1} K_{\frac{p}{n+1}}\left(\frac{E^{n+1}}{n+1}\right) K_{\frac{q}{n+1}}\left(\frac{E^{n+1}}{n+1}\right).$$

Then \mathbf{Q}_{CFIV} is expressed as

$$\mathbf{Q}_{\text{CFIV}} = \frac{8}{\pi(k+2)} (\mathcal{G}_{\frac{k}{2}}(\frac{1}{2}, \frac{1}{2}, \frac{k}{2} - 1) - \mathcal{G}_{\frac{k}{2}}(\frac{1}{2}, -\frac{k+1}{2}, 0) - \mathcal{G}_{\frac{k}{2}}(\frac{1}{2}, \frac{k+3}{2}, 0)).$$

On the other hand, $\mathcal{G}_n(p, q, \ell)$ is already evaluated in [39],

$$\begin{aligned} \mathcal{G}_n(p, q, \ell) = & \frac{(2(n+1))^{1+\frac{\ell}{n+1}}}{4\Gamma(2+\frac{\ell}{n+1})} \Gamma(1+\frac{p+q+\ell}{2(n+1)}) \Gamma(1+\frac{-p+q+\ell}{2(n+1)}) \\ & \times \Gamma(1+\frac{p-q+\ell}{2(n+1)}) \Gamma(1+\frac{-p-q+\ell}{2(n+1)}). \end{aligned} \quad (93)$$

Substituting these results, we find that the resultant \mathbf{Q}_{CFIV} is simplified considerably,

$$\mathbf{Q}_{\text{CFIV}} = \frac{k}{k+2}$$

which agrees with the known result in [15].

5.2 Sub-leading perturbations

In [19], sub-leading perturbation potentials are also discussed, for example,

$$W(z, t) = \frac{z^6}{6} - \frac{tz^2}{2}$$

which is simply rewritten as $t^{3/2}W_1(z^2/t^{1/2})/2$. In the point of view of functional integrals over super-fields, it is argued that \mathbf{Q}_{CFIV} must be twice of that for $W_1(z)$. Let us interpret this in terms of an ODE.

According to our working hypothesis, consider

$$\psi(t) = \int_c e^{W(z,t)} dz.$$

It satisfies the 3rd order ODE

$$\frac{d^3}{dt^3}\psi(t) = \frac{1}{8}\left(\psi(t) + 2t\frac{d}{dt}\psi(t)\right)$$

or in terms of a rescaled variable $x = t/(2)^{4/3}$,

$$\frac{d^3}{dx^3}\phi(x) - 4x\frac{d}{dx}\phi(x) - 2\phi(x) = 0$$

where $\phi(x) = \psi(2^{4/3}x)$.

It is well-known that $\text{Ai}^2(x)$, $\text{Bi}^2(x)$ and $\text{Ai}(x)\text{Bi}(x)$ are solutions to this ODE. Now we shift $x \rightarrow x - E$ and take ϕ_j^2, ϕ_{j+1}^2 and $\phi_j\phi_{j+1}$ as FSS in \mathcal{S}_j where ϕ_j is defined in (4). Indeed, it is easily checked that

$$W[\phi_j^2, \phi_{j+1}^2, \phi_j\phi_{j+1}] = 2(W[\phi_j, \phi_{j+1}])^2$$

where W denotes the Wronskian determinant. Thus three functions are linearly independent. The connection between \mathcal{S}_0 and \mathcal{S}_{-1} is easily solved. It follows from the three term relation of the Ai function that

$$\phi_1^2 = \phi_0^2 + \phi_{-1}^2 + 2\phi_0\phi_{-1} = \phi_0^2 + \phi_2^2 + 2\phi_0\phi_2.$$

With nonzero angular momentum, this may be modified as

$$T\phi_1^2 = \phi_0^2 + \phi_{-1}^2 + 2\phi_0\phi_{-1} = \phi_0^2 + \phi_2^2 + 2\phi_0\phi_2.$$

The Stokes multiplier is thus squared $T = \tau_1^2$. Since \mathbf{Q}_{CFIV} is essentially the logarithm of the Stokes multiplier, this means \mathbf{Q}_{CFIV} should be doubled, in agreement with the argument in [19].

More generally, for a perturbation potential

$$W_m(z, t) = \frac{z^{2m}}{2m} - \frac{t}{2}z^2,$$

the associated function

$$\psi_m(t) = \int_{\mathcal{C}} e^{W_m(z, t)} dz$$

is found to satisfy a special case of $\text{so}(m+1)$ ODE in the classification of [27],

$$\frac{d^m}{dt^m} \psi_m(t) = (-1)^m \sqrt{t} \frac{d}{dt} \sqrt{t} \psi_m(t).$$

This is again consistent with the observation in [19].

5.3 Eigenvalues of conserved quantities

The formula for the vacuum expectation values of conserved quantities \mathbf{I}_{2n-1} in CFT based on $U_q(sl_2)$ symmetry is conjectured in [24]. In the present framework, the result is translated to

$$\int e^{m\theta} \ln(1 + \mathbf{Y}_k(\theta)) \frac{d\theta}{2\pi} \propto \mathbf{I}_m$$

for $k = 1$ and $m = 2n - 1$. The analytic evaluation of the above is difficult except for the $m = 1$ case to which the dilogarithm technique can be applied. This is due to the fact that the analytic expression of Y_k is not available in general.

Although all \mathbf{I}_m become null strictly at $h = 0$, we expect the left hand side brings conserved quantities order by order in h . Indeed, the first order

quantity in h for $m = 1$ agrees with the CFIV index. Let us assume that this conjecture is valid for arbitrary k and m and evaluate

$$\tilde{\mathbf{I}}_m := \int e^{m\theta} e^{-A(\theta)} \frac{d\theta}{2\pi}.$$

Since we have the explicit solution to $e^{-A(\theta)}$, the evaluation of $\tilde{\mathbf{I}}_m$ is immediate. The actual calculation goes parallel to the one for the CFIV index. Thanks to (93), we find

$$\tilde{\mathbf{I}}_m = \frac{2^{2(m-1)}}{\pi} \frac{\Gamma(\frac{m}{2})^2}{\Gamma(m)} \frac{\Gamma(\frac{m}{2} + \frac{1}{k+2}) \Gamma(\frac{m}{2} + 1 - \frac{1}{k+2})}{\Gamma(\frac{1}{2} + \frac{1}{k+2}) \Gamma(\frac{1}{2} - \frac{1}{k+2})}.$$

6 Summary and conclusion

In this report, we propose a hypothesis that the combination of the wave functions associated with F -term potentials yields the Y functions of the corresponding massless TBA equations. This has been successfully demonstrated for nontrivial examples. As applications, expectation values of conserved quantities, beyond the CFIV index, are conjectured by using the explicit solutions.

There remain obviously many questions open.

The unexpected cancellation and the truncation of TBA equations, observed for the $SU(3)$ case, may be generic for $SU(n \geq 3)$, which needs a proof. Moreover, we need to understand the intrinsic reason why such a miraculous cancellation should occur.

The super-potentials for the super-conformal theories, except for A type, contain multi-variables [17, 18]. It is not clear how to extend the observation in this report, especially how to define “wave functions”, in these cases.

The argument given in this communication is restricted to the massless case, in which ODE equations help us to find solutions. The original problem is concerned, however, with the generally massive quantum field theory: the TBA, or the CFIV index are developed for the analysis of such a case. We note the recent progress in the ODE/IM correspondence towards the massive deformation [40, 41, 42, 43]. Hopefully it will help us to analyze the massive TBA through auxiliary linear problems associated to integrable partial differential equations.

Ultimately, the reason why our hypothesis works well remains a mystery. This is the most serious problem at the present moment. We hope to come back to this in the near future.

Acknowledgements

The author would like to thank P. Dorey, A. Klümper and R. Tateo for the critical reading of the manuscript. He also thanks for the warm hospitalities by the organizers of “infinite analysis 2014” and “integrable lattice models and quantum field theories” where a part of the present contents was presented. This work has been supported by JSPS Grant-in-Aid for Scientific Research (C) No. 24540399.

Appendix A Dressed Vacuum Form : SU(3)

We write down (72) in a way that the connection to integrable systems is obvious. For this purpose, we introduce

$$D^{(2)}(E, \mathbf{g}) = \left(\prod_{0 \leq i < j \leq 2} (g_j - g_i) \right) (D^{[0]}(E) D^{[1]}(E\Omega^3) q^{-g_1} - D^{[1]}(E) D^{[0]}(E\Omega^3) q^{-g_0}).$$

By using the elementary formula on matrix determinants [34], we can derive,

$$\tau_1^{(1)}(E) = \xi_0 \frac{D^{[0]}(E)}{D^{[0]}(E\Omega^3)} + \xi_1 \frac{D^{[0]}(E\Omega^6)}{D^{[0]}(E\Omega^3)} \frac{D^{(2)}(E)}{D^{(2)}(E\Omega^3)} + \xi_2 \frac{D^{(2)}(E\Omega^6)}{D^{(2)}(E\Omega^3)},$$

where we set

$$\xi_0 = q^{g_0-1}, \quad \xi_1 = q^{g_1-1}, \quad \xi_2 = q^{g_2-1}.$$

Further, use $\Omega^3 = q^{-1}$ and write

$$\mathbf{Q}(E) = D^{[0]}(E), \quad \bar{\mathbf{Q}}(E) = D^{(2)}(Eq^{\frac{1}{2}}), \quad T_1^{(1)}(E) = \tau_1^{(1)}(Eq^{\frac{5}{4}}),$$

then we obtain the expression,

$$\mathbf{T}_1^{(1)}(E) = \xi_0 \frac{\mathbf{Q}(Eq^{\frac{5}{4}})}{\mathbf{Q}(Eq^{\frac{1}{4}})} + \xi_1 \frac{\mathbf{Q}(Eq^{-\frac{3}{4}})}{\mathbf{Q}(Eq^{\frac{1}{4}})} \frac{\bar{\mathbf{Q}}(Eq^{\frac{3}{4}})}{\bar{\mathbf{Q}}(Eq^{-\frac{1}{4}})} + \xi_2 \frac{\bar{\mathbf{Q}}(Eq^{-\frac{5}{4}})}{\bar{\mathbf{Q}}(Eq^{-\frac{1}{4}})}.$$

This agrees with the known DVF for the integral model with $A_2^{(1)}$ symmetry. Similarly, by setting $T_1^{(2)}(E) = \tau_1^{(2)}(Eq^{\frac{7}{4}})$ one finds,

$$\mathbf{T}_1^{(2)}(E) = \xi'_0 \frac{\bar{\mathbf{Q}}(Eq^{\frac{5}{4}})}{\bar{\mathbf{Q}}(Eq^{\frac{1}{4}})} + \xi'_1 \frac{\bar{\mathbf{Q}}(Eq^{-\frac{3}{4}})}{\bar{\mathbf{Q}}(Eq^{\frac{1}{4}})} \frac{\mathbf{Q}(Eq^{\frac{3}{4}})}{\mathbf{Q}(Eq^{-\frac{1}{4}})} + \xi'_2 \frac{\mathbf{Q}(Eq^{-\frac{5}{4}})}{\mathbf{Q}(Eq^{-\frac{1}{4}})}$$

where

$$\xi'_0 = \xi_0 \xi_1 = (\xi_2)^{-1}, \quad \xi'_1 = (\xi_1)^{-1}, \quad \xi'_2 = (\xi_0)^{-1}.$$

Appendix B Solutions to the 3rd order ODE

There are elementary functions that solve the 3rd order equations, but are not widely known in contrast to the 2nd order case. It is shown, however, in [37] that the solution $\varphi(x)$ to (65) with the desired asymptotic behavior (2) is given by Meijer's G function⁶,

$$\varphi(x) = c G_{0,3}^{3,0} \left(- \middle| \frac{x^4}{4^3} \right)_{0, \frac{1}{4}, \frac{2}{4}}$$

for $0 \leq \arg(x^4/4^3) \leq 6\pi$. The normalization constant c should be determined so as to be consistent with (2). To verify this, one requires several machineries.

In this appendix, we take a formal but simple approach to represent the solution $\varphi(x)$: we try to represent it with a familiar object, a limit of the generalized hypergeometric series ${}_3F_2$. The following argument generalizes the one given for the Airy function in its relation to finite size lattice models [38].

The generalized hypergeometric series ${}_3F_2(a_1, a_2, a_3; b_1, b_2 | \zeta)$ satisfies the following differential equation,

$$[\delta(\delta + b_1 - 1)(\delta + b_2 - 1) - z(\delta + a_1)(\delta + a_2)(\delta + a_3)]{}_3F_2 = 0, \quad (\text{B.94})$$

$$\delta := \zeta \frac{d}{d\zeta}.$$

In the following we adopt abbreviations $\mathbf{a} := (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2)$ and ${}_3F_2(\mathbf{a}; \mathbf{b} | \xi)$.

Consider the limit

$$\lim_{N \rightarrow \infty} {}_3F_2 \left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| \frac{x^L}{L^3 N(N^2 - 1)} \right).$$

It is easy to check that it satisfies (65) where $L = 4$, $\sigma = -\frac{1}{4}$ and

$$\mathbf{a} = (-N, -N + \sigma, -N + 2\sigma), \quad \mathbf{b} = (1 + \sigma, 1 + 2\sigma).$$

The parameter N corresponds to the lattice size for the SU(2) case [38].

The desired solution, φ , however should satisfy the asymptotic behavior (2). In order to accomplish this, we first keep N a large but finite positive integer and use the fact that two other independent solutions to (B.94) are

$$\zeta^{-\sigma} {}_3F_2(\mathbf{a}'; \mathbf{b}' | \zeta) \quad \text{and} \quad \zeta^{-2\sigma} {}_3F_2(\mathbf{a}''; \mathbf{b}'' | \zeta)$$

⁶to meet (65), we rotate $x \rightarrow e^{i\pi/4}x$.

where $\mathbf{a}' = \mathbf{a} - \sigma(1, 1, 1)$, $\mathbf{b}' = (1 - \sigma, 1 + \sigma)$ and $\mathbf{a}'' = \mathbf{a} - 2\sigma(1, 1, 1)$, $\mathbf{b}'' = (1 - \sigma, 1 - 2\sigma)$.

Let a subsidiary variable $z^L = \zeta$ and a subsidiary function $\Psi_{N,\sigma}$ be

$$\Psi_{N,\sigma}(x) = \frac{z^{L\sigma+1}}{(1 - z^L)^N} {}_3F_2(\mathbf{a}; \mathbf{b} | z^L).$$

The ODE for $\Psi_{N,\sigma}$ reads,

$$\begin{aligned} & z^3 \frac{d^3}{dz^3} \Psi_{N,\sigma} - L^2 \frac{3N(N+1)z^L}{(z^L - 1)^2} z \frac{d}{dz} \Psi_{N,\sigma} \\ & + L^3 \frac{N(N^2 - 1)(z^L + 1)z^L}{(1 - z^L)^3} \Psi_{N,\sigma} + 3L^2 \frac{N(N+1)z^L}{(1 - z^L)^2} \Psi_{N,\sigma} = 0, \end{aligned} \quad (\text{B.95})$$

where $\sigma L = -1$ is used .

We select a solution which vanishes at the regular singular point $z = 1$. Let us consider the linear combination of three independent solutions

$$\Psi_{N,\sigma} := \frac{1}{(1 - z^L)^N} (\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3)$$

where

$$\begin{aligned} \psi_1 &= z^{L\sigma+1} {}_3F_2(\mathbf{a}; \mathbf{b}; z^L), \\ \psi_2 &= z {}_3F_2(\mathbf{a}'; \mathbf{b}'; z^L), \\ \psi_3 &= z^{-L\sigma+1} {}_3F_2(\mathbf{a}''; \mathbf{b}''; z^L). \end{aligned}$$

The characteristic exponents at $z = 1$ of (B.95) are $-N, -N + 1, 2N + 2$, thus it suffices to require

$$(\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3)|_{z=1} = 0, \quad \frac{d}{dz}(\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3)|_{z=1} = 0.$$

These fix α_i (but for an overall factor),

$$\alpha_1(N) = c \frac{W[\psi_2, \psi_3]}{W[\psi_1, \psi_2, \psi_3]}, \quad \alpha_2(N) = -c \frac{W[\psi_1, \psi_3]}{W[\psi_1, \psi_2, \psi_3]}, \quad \alpha_3(N) = c \frac{W[\psi_1, \psi_2]}{W[\psi_1, \psi_2, \psi_3]}$$

where W denotes the Wronskian determinant evaluated at $z = 1$.

We assume the following limit exists,

$$\alpha_i = \lim_{N \rightarrow \infty} \left(\frac{1}{4^3 N(N^2 - 1)} \right)^{\frac{i-1}{4}} \alpha_i(N)$$

although the actual evaluation of the limit may not be simple.

In the scaled variables x , the regular singular point is relocated at $x_0 = (4^3 N(N^2 - 1))^{\frac{1}{4}}$. By an argument similar to the $SU(2)$ case, we can show that $\Psi_{N,\sigma}$ is a decreasing function in $0 < x < x_0$. In the “scaling limit” $N \rightarrow \infty$, x_0 also goes to infinity. Thus we conclude that $\varphi = \lim_{N \rightarrow \infty} \Psi_{N,\sigma}$ is the desired decaying function, which is written as follows,

$$\varphi(x) = \alpha_1 {}_0F_2\left(\frac{3}{4}, \frac{1}{2} \middle| \frac{x^4}{4^3}\right) + \alpha_2 x {}_0F_2\left(\frac{3}{4}, \frac{5}{4} \middle| \frac{x^4}{4^3}\right) + \alpha_3 x^2 {}_0F_2\left(\frac{5}{4}, \frac{3}{2} \middle| \frac{x^4}{4^3}\right).$$

Appendix C The cancellation of terms

Using the expansions (81) into definitions of (77), we obtain the expressions of $y_{j,m}^{(a)}(\theta)$ in terms of $t_{j',m'}^{(a')}(\theta)$. Especially we are interested in $y_{2,2}^{(1)}(\theta) + y_{2,3}^{(2)}(\theta)$ and $y_{2,3}^{(1)}(\theta) + y_{2,2}^{(2)}(\theta)$. Since $\mathbf{T}_m^{(a)}$ satisfies the T-system, $t_{j',m'}^{(a')}(\theta)$ are not necessarily independent: there are some relations, e.g.,

$$\begin{aligned} t_{1,3}^{(1)}(\theta) &= -t_{1,2}^{(2)}(\theta), & t_{1,3}^{(2)}(\theta) &= -t_{1,2}^{(1)}(\theta), \\ t_{1,4}^{(1)}(\theta) &= -t_{1,1}^{(2)}(\theta), & t_{1,4}^{(2)}(\theta) &= -t_{1,1}^{(1)}(\theta), \\ t_{2,3}^{(1)}(\theta) &= t_{1,2}^{(1)}(\theta + i\frac{\pi}{3}) + t_{1,2}^{(1)}(\theta - i\frac{\pi}{3}) + t_{1,1}^{(1)}(\theta)t_{1,2}^{(2)}(\theta) - t_{2,2}^{(2)}(\theta), \\ t_{2,3}^{(2)}(\theta) &= t_{1,2}^{(2)}(\theta + i\frac{\pi}{3}) + t_{1,2}^{(2)}(\theta - i\frac{\pi}{3}) + t_{1,1}^{(2)}(\theta)t_{1,2}^{(1)}(\theta) - t_{2,2}^{(1)}(\theta) \end{aligned}$$

and so on. Using these relations, we find relatively simple expressions,

$$\begin{aligned} y_{2,3}^{(1)}(\theta) + y_{2,2}^{(2)}(\theta) &= (t_{1,1}^{(2)}(\theta))^2 - \left(\frac{t_{1,2}^{(2)}(\theta + i\frac{\pi}{3})t_{1,2}^{(2)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)}\right)^2 - t_{2,4}^{(1)}(\theta) - t_{2,1}^{(2)}(\theta), \\ y_{2,2}^{(1)}(\theta) + y_{2,3}^{(2)}(\theta) &= (t_{1,1}^{(1)}(\theta))^2 - \left(\frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(2)}(\theta)}\right)^2 - t_{2,1}^{(1)}(\theta) - t_{2,4}^{(2)}(\theta). \end{aligned}$$

They however still show that we need $O(h^2)$ terms $t_{2,m}^{(a)}$ in the expansions of $\mathbf{T}_m^{(a)}$. Note that

$$y_{1,2}^{(2)}(\theta) = \frac{t_{1,2}^{(2)}(\theta + i\frac{\pi}{3})t_{1,2}^{(2)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)} \quad \text{and} \quad y_{1,2}^{(1)}(\theta) = \frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(2)}(\theta)} \quad (\text{C.96})$$

also appear in the last term of (87).

The key idea is to employ the expansions of $D^{[i]}(E, \mathbf{g})$,

$$D^{[i]}(E, \mathbf{g}) = d_0^{[i]}(E) + h d_1^{[i]}(E) + h^2 d_2^{[i]}(E) + O(h^3), \quad (i = 0, 1, 2).$$

Although $d_0^{[i]}(E)$ is denoted by $Q^{[i]}(E)$ in the main body, we use $d_0^{[i]}$ in the appendix for uniformity. Substituting this into (73), taking account of the shift in (74) and (75), we are able to derive the expressions of $t_{j,m}^{(a)}$ in terms of $d_j^{[i]}$. The point is that the latter has a smaller number of independent elements (for fixed order of expansions in h) and a smaller number of relations.

The obvious constraint is $\tau_0^{(1)}(E, \mathbf{g}) = 1$. We write it as

$$\tau_0^{(1)}(E, \mathbf{g}) = 1 = \tau_{0,0}^{(1)}(E) + h \tau_{1,0}^{(1)}(E) + h^2 \tau_{2,0}^{(1)}(E) + O(h^2)$$

or

$$\tau_{0,0}^{(1)}(E) = 1, \quad \tau_{1,0}^{(1)}(E) = 0, \quad \tau_{2,0}^{(1)}(E) = 0 \dots$$

The constraints above are written in terms of sums of triple products of $d_j^{[i]}$. By comparing the expressions in terms of $d_j^{[i]}$, we find

$$\begin{aligned} t_{1,1}^{(2)}(\theta) + \frac{t_{1,2}^{(2)}(\theta + i\frac{\pi}{3})t_{1,2}^{(2)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)} &= -i\tau_{0,0}^{(1)}(-Ee^{\frac{3}{8}\pi}) = -i, \\ t_{1,1}^{(1)}(\theta) + \frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(2)}(\theta)} &= i\tau_{0,0}^{(1)}(-Ee^{\frac{5}{8}\pi}) = i. \end{aligned}$$

There are also unexpected relations,

$$\begin{aligned} t_{1,1}^{(2)}(\theta) - \frac{t_{1,2}^{(2)}(\theta + i\frac{\pi}{3})t_{1,2}^{(2)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)} + i\tau_{0,0}^{(1)}(-Ee^{\frac{3}{8}\pi}) \\ = t_{1,1}^{(2)}(\theta) - \frac{t_{1,2}^{(2)}(\theta + i\frac{\pi}{3})t_{1,2}^{(2)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(1)}(\theta)} + i \\ = 6(-d_0^{[0]}(E\omega^7)d_0^{[1]}(E\omega^3) + id_0^{[0]}(E\omega^3)d_0^{[1]}(E\omega^7))d_0^{[2]}(-E\omega^3), \\ t_{1,1}^{(1)}(\theta) - \frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(2)}(\theta)} - i\tau_{0,0}^{(1)}(-Ee^{\frac{5}{8}\pi}) \\ = t_{1,1}^{(1)}(\theta) - \frac{t_{1,2}^{(1)}(\theta + i\frac{\pi}{3})t_{1,2}^{(1)}(\theta - i\frac{\pi}{3})}{t_{1,2}^{(2)}(\theta)} - i \\ = 6(id_0^{[0]}(-E\omega^5)d_0^{[1]}(-E\omega) + d_0^{[0]}(-E\omega)d_0^{[1]}(-E\omega^5))d_0^{[2]}(E\omega^5) \end{aligned}$$

where $\omega = e^{\frac{\pi}{8}i}$. We also find

$$\begin{aligned}
& t_{2,1}^{(2)}(\theta) + t_{2,4}^{(1)}(\theta) + 1 \\
&= t_{2,1}^{(2)}(\theta) + t_{2,4}^{(1)}(\theta) + \tau_{0,0}^{(1)}(-E\omega^3) - 2\tau_{2,0}^{(1)}(-E\omega^3) \\
&= -3(d_0^{[0]}(E\omega^3)d_0^{[1]}(E\omega^7) + id_0^{[0]}(E\omega^7)d_0^{[1]}(E\omega^3))d_0^{[2]}(-E\omega^3), \\
& t_{2,1}^{(1)}(\theta) + t_{2,4}^{(2)}(\theta) + 1 \\
&= t_{2,1}^{(1)}(\theta) + t_{2,4}^{(2)}(\theta) + \tau_{0,0}^{(1)}(-E\omega^5) - 2\tau_{2,0}^{(1)}(-E\omega^5) \\
&= 3(d_0^{[0]}(-E\omega^5)d_0^{[1]}(-E\omega) - id_0^{[0]}(-E\omega)d_0^{[1]}(-E\omega^5))d_0^{[2]}(E\omega^5).
\end{aligned}$$

Using these results, the sums of our interest simplify considerably,

$$\begin{aligned}
y_{2,3}^{(1)}(\theta) + y_{2,2}^{(2)}(\theta) &= 9(d_0^{[0]}(E\omega^3)d_0^{[1]}(E\omega^7) + id_0^{[0]}(E\omega^7)d_0^{[1]}(E\omega^3))d_0^{[2]}(-E\omega^3), \\
y_{2,2}^{(1)}(\theta) + y_{2,3}^{(2)}(\theta) &= 9(id_0^{[0]}(-E\omega)d_0^{[1]}(-E\omega^5) - d_0^{[0]}(-E\omega^5)d_0^{[1]}(-E\omega))d_0^{[2]}(E\omega^5).
\end{aligned}$$

The right hand sides of (C.96) are also derived from the above results,

$$\begin{aligned}
y_{1,2}^{(2)}(\theta) &= 3(d_0^{[0]}(E\omega^7)d_0^{[1]}(E\omega^3) - id_0^{[0]}(E\omega^3)d_0^{[1]}(E\omega^7))d_0^{[2]}(-E\omega^3), \\
y_{1,2}^{(1)}(\theta) &= -3(d_0^{[0]}(-E\omega)d_0^{[1]}(-E\omega^5) + id_0^{[0]}(-E\omega^5)d_0^{[1]}(-E\omega))d_0^{[2]}(E\omega^5).
\end{aligned} \tag{C.97}$$

Thereby, we conclude the validity of (88).

References

- [1] M. Takahashi, “Thermodynamics of One-Dimensional Solvable Models”, Cambridge University Press, (1999).
- [2] Al. B. Zamolodchikov, “Thermodynamic Bethe ansatz in relativistic models: Scaling 3-state Potts and Lee-Yang models”, Nucl. Phys. **B 342** (1990) 695-720.
- [3] P. Dorey, A. Pocklington and R. Tateo, “Integrable aspects of the scaling q-state Potts models I: bound states and bootstrap closure”, Nucl. Phys. **B 661** (2003) 425-463.
- [4] M. Gaudin, “Thermodynamics of the Heisenberg-Ising Ring for $\Delta \geq 1$ ”, Phys. Rev. Lett. **26** (1971) 1301.
- [5] M. Takahashi and M. Suzuki, “One-Dimensional Anisotropic Heisenberg Model at Finite Temperatures”, Prog. Theoret. Phys.**48** (1972) 2187-2209.
- [6] A. Klümper, “Free energy and correlation lengths of quantum chains related to restricted solid-on-solid lattice models”, Ann. Phys.**504** (1992) 540-553.
- [7] G. Jüttner, A. Klümper and J. Suzuki, “From fusion hierarchy to excited state TBA”, Nucl. Phys. **B 512** (1998) 581-600.
- [8] P. Dorey, I. Runkel, R. Tateo and G. Watts, “g-function flow in perturbed boundary conformal field theories ”, Nucl. Phys. **B578** (2000) 85-122
- [9] A. M. Tsvelick and P. B. Wiegmann, “ Exact results in the theory of magnetic alloys”, Adv. in Phys. **32** (1983) 453-713.
- [10] P. Fendley, “Airy functions in thermodynamic Bethe ansatz”, Letters in Math. Phys. **49** (1999) 229-233.
- [11] B. M. McCoy, C. Tracy and T. T. Wu, “Painlevé functions of the third kind”, J. Math. Phys. **18** (1977) 1058.
- [12] Al. B. Zamolodchikov, “Painlevé III and 2D polymers”, Nucl. Phys. **B 432** (1994) 427-456.
- [13] C. A. Tracy, H. Widom, “Proofs of two conjectures related to the thermodynamic Bethe ansatz”, Comm. Math. Phys. **179** (1996) 667-680.

- [14] P. Fendley and K. Intriligator, “Scattering and thermodynamics of fractionally-charged supersymmetric solitons”, Nucl. Phys. **B 372** (1992) 533-558.
- [15] S. Cecotti, P. Fendley, K. Intriligator and C. Vafa “A new supersymmetric index”, Nucl. Phys. **B 386** (1992) 405-452.
- [16] P. Dorey, T. C. Dunning, R. Tateo “The ODE/IM Correspondence”, J. Phys. **A40** (2007) R205.
- [17] C. Vafa and N. P. Warner “Catastrophes and the Classification of Conformal Theories ”, Phys. Lett. **B218** (1989) 51.
- [18] E. J. Martinec, “Algebraic Geometry and Effective Lagrangians ”, Phys. Lett. **B217** (1989) 431.
- [19] S. Cecotti and C. Vafa, “Topological-anti-topological fusion”, Nucl. Phys. **B 367** (1991) 359-461.
- [20] P. Dorey, R. Tateo, “Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equations”, J. Phys. **A32** (1999) L419.
- [21] P. Dorey, R. Tateo, “On the relation between Stokes multipliers and the T-Q systems of conformal field theory”, Nucl. Phys. **B 563** (1999) 573-602.
- [22] R. Baxter, “Exactly solved models in statistical mechanics”, Academic Press (1982) .
- [23] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, “Spectral determinants for Schrödinger equation and Q-operators of conformal field theory”, J. Stat. Phys. **102**(2001) 567-576.
- [24] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, “Integrable Structure of Conformal Field Theory II. Q-operator and DDV equation”, Comm. Math. Phys. **190**(1997) 247-278.
- [25] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, “Integrable structure of conformal field theory, quantum KdV theory and Thermodynamic Bethe Ansatz”, Comm. Math. Phys. **177**(1996) 381-398.
- [26] S. L. Lukyanov, “Notes on parafermionic QFT’s with boundary interaction”, Nucl. Phys. **B 784** (2007) 151-201.

- [27] P. Dorey, T. C. Dunning, D. Masoero, J. Suzuki and R. Tateo, “Pseudo-differential equations, and the Bethe ansatz for the classical Lie algebras”, Nucl. Phys. **B 772** (2007) 249-289.
- [28] A. Klümper and P. Pearce, “Conformal weights of RSOS lattice models and their fusion hierarchies”, Physica **A 183** (1992) 304.
- [29] A. Kuniba, T. Nakanishi and J. Suzuki, “Functional relations in solvable lattice models I: functional relations and representation theory ”, Int. J. Mod. Phys., **A9** (1994) 5215.
- [30] Al. B. Zamolodchikov, “On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories”, Phys. Lett. **B253** (1991) 391-394.
- [31] Al. B. Zamolodchikov, “ From tricritical Ising to critical Ising by thermodynamic Bethe ansatz ”, Nucl. Phys. **B 358** (1991) 524- 546.
- [32] P. Dorey and R. Tateo, “Differential equations and integrable models: the SU(3) case”, Nucl. Phys. **B 571** (2001) 583-606.
- [33] V. V. Bazhanov, A. N. Hibbard and S. M. Khoroshkin, “Integrable structure of W_3 Conformal Field Theory, Quantum Boussinesq Theory and Boundary Affine Toda Theory”, Nucl. Phys. **B 622** (2002) 475-547.
- [34] P. Dorey, T. C. Dunning and R. Tateo, “Differential equations for general SU(n) Bethe ansatz systems”, J. Phys. **A33** (2000) 8427-8441.
- [35] J. Suzuki, “Functional relations in Stokes multipliers and solvable models related to $U_q(A_n^{(1)})$ ”, J. Phys. **A33** (2000) 3507-3521.
- [36] J. Damerau, “Nonlinear integral equations for the thermodynamics of integrable quantum chains”, (Thesis, Wuppertal University 2008)
- [37] B. L. J. Braaksma, “Asymptotic Analysis of Differential Equation of Turrittin”, SIAM J. Math. Anal. **2** (1971) 1-16.
- [38] P. Dorey, J. Suzuki and R. Tateo, “Finite lattice Bethe ansatz systems and the Heun equation”, J. Phys. **A37** (2004) 2047.
- [39] S. Cecotti, L. Girardello and A. Pasquinucci, “Non-perturbative aspects and exact results for $\mathcal{N} = 2$ Landau-Ginzburg models”, Nucl. Phys. **B 328** (1989) 701-722.

- [40] S. L. Lukyanov, A. B. Zamolodchikov, “Quantum sine(h)-Gordon model and classical integrable equations”, JHEP **07** (2010)008.
- [41] P. Dorey, S. Faldella, S. Negro and R. Tateo, “The Bethe ansatz and the Tzitzéica-Bullough-Dodd equation”, Phil. Trans. R. Soc. **A 371** 20120052.
- [42] K. Ito and C. Locke, “ODE/IM correspondence and modified affine Toda field equations”, Nucl. Phys. **B885** (2014) 600-619(arXiv:1312.6759).
- [43] P. Adamopoulou and C. Dunning, “Bethe Ansatz equations for the classical $A_n^{(1)}$ affine Toda field theories”, J. Phys. **A47** (2014) 205205 (arXiv:1401.1187).